The Role of Dice in World Domination

A Case Study of Risk: The Board Game

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Izleti v matematično vesolje

The classic board games

Then came the 'modern' classics

And things escalated quickly

Enter: Risk


PARKER
RIZIKÓ
TÖBB
MINT
300
FIGURÁVAL
Világhódítók játéka

A game of epic proportions

A game of epic proportions

A quick rules overview

- The map is divided into territories (grouped into continents).
- On your turn: **reinforce** $\rightarrow$ **attack** $\rightarrow$ **fortify**.
- You can attack only neighboring territories.
- Battles are dice-based:
  - attacker rolls up to 3 dice (depending on number of attacking armies),
  - defender rolls up to 2 dice (depending on number of attacking armies),
  - highest and second highest dice rolls are compared; loser(s) lose armies.

How can we represent a game?

A broad mathematical viewpoint:

- **State** = a complete snapshot of the game right now.
- **Action** = a legal move a player can choose.
- **Transition rule** = what the rules do when an action happens.

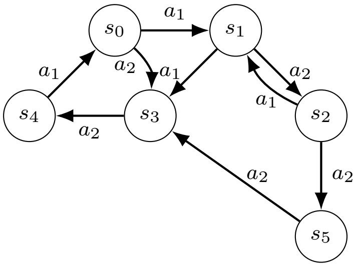
$$\text{Game} = (\mathcal{S}, \mathcal{A}, T)$$

$$T(s, a) \rightarrow \text{next state}(s)$$

State space graph

- Each node is a **state** (s_i) of the game.
- Directed edges are **actions** (a_j) a player can take.
- Playing the game means choosing actions and moving through this graph.

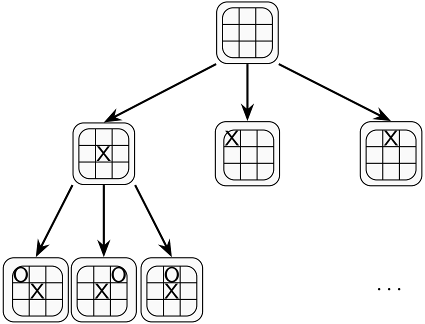
Great idea... until you realize how many states there are.



Toy state space graph

Example partial game tree: tic-tac-toe

- Root: empty board
- First branching: where does **X** play?
- Second branching: where does **O** reply?



Why this matters

For small games, computers can explore (nearly) the whole tree.

Chess and Go are already hard

- Branching factor (b): number of choices at each step.
- Plies (d): in how many steps does the game end.
- Total nodes explored grows **exponentially**.

$$\text{Game-tree complexity} \geq b^d$$

Game	**Board size (positions)**	**State-space complexity ($\log_{10}$)**	**Game-tree complexity ($\log_{10}$)**	**Average game length (plies)**	**Branching factor**
Tic-tac-toe	9	3	5	9	4
Connect Four	42	12	21	36	4
Chess	64	44	123	70	35
Go (19×19)	361	170	505	211	250

Tic-tac-toe: State Space and Game Tree Reduction

- **State-space upper bound:** 9 cells has 3 states each (X, O, empty), $3^9 = 19{,}683$ possible boards.
- **Removing illegal positions:** 5,478 legal positions.
- **Considering rotations and reflections**: only 765 different positions.
- **Game-tree bound:** if all games would last 9 plies, $9! = 362{,}880$ possible sequences.
- **Games end earlier:** 255,168 possible games.
- **Symmetry in the game tree:** 26,830 different games.

Briefly: major search strategies

How do computers play games?

- Look ahead a few moves.
- Score future positions using a heuristic.
- Choose the move that maximizes expected success.

Common buzzwords:

- Minimax algorithm
  - Recursive algorithm for evaluating game state
  - Can heuristically evaluate non-final states, e.g. Deep Blue
- Alpha–beta pruning
  - Skipping bad branches
- Monte Carlo Tree Search
  - Random sampling of the search space

What makes Risk even harder?

Risk adds multiple layers of complexity:

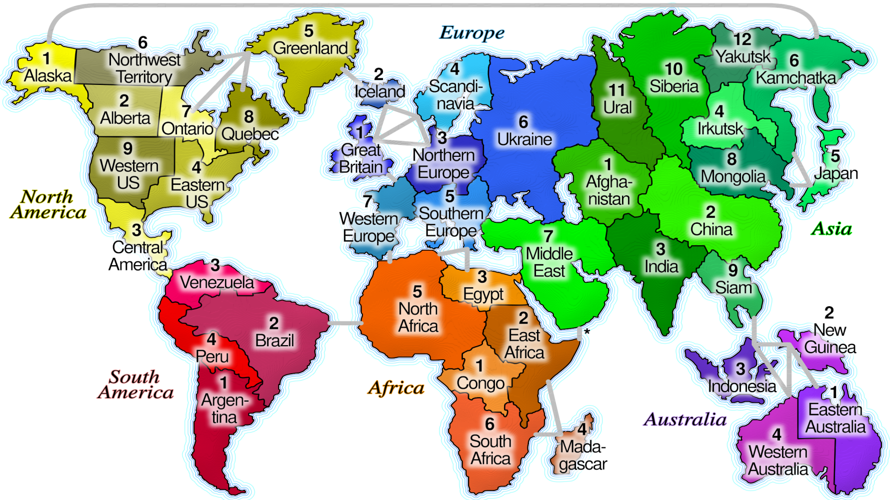
- Multi-player
- Huge state description
  - Territory ownership and troop counts
- **Randomness of dice at every battle**

How to represent all of this?

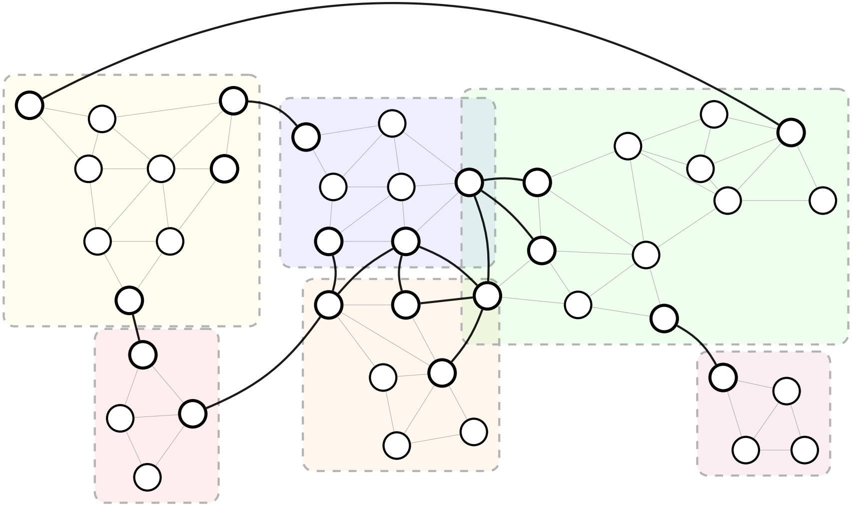
The Risk map is a graph

- Territories = nodes
- Borders = edges (legal attacks)
- Continents = subgraphs
- Some nodes are strategically special:
  - **chokepoints** (few entry edges),
  - **bridges** between continents,
  - **high-degree hubs** (many neighbors).

Territories on the Risk map



Risk map is a graph



Topology of the map

A continent's "value" depends on how many reinforcement units you get *for controlling the entire continent*.

Continent	Bonus	Territories	Border territory	Bonus / border	Bonus / territory
Australia	+2	4	1	2.00	0.50
North America	+5	9	3	1.67	0.56
Asia	+7	12	5	1.40	0.58
Europe	+5	7	4	1.25	0.71
South America	+2	4	2	1.00	0.50
Africa	+3	6	3	1.00	0.50

Comparison of reinforcement details for each continent.

While Australia seems to be a good choice, it is actually agreed that North America is the most important.

Units and control represented on the graph

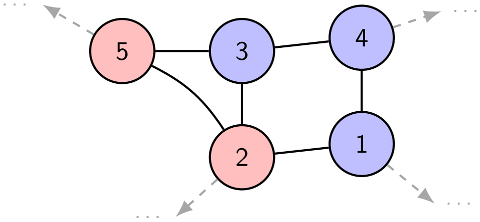
A game state can include in the topology graph:

- who owns each territory (a color),
- how many troops are on each territory (a number).

Posible actions change these values on the graph.

- Reinforcements: increase numbers on selected nodes.
- Attacks: change numbers (losses) and sometimes flip ownership (color).
- Fortify: move numbers along edges you control.

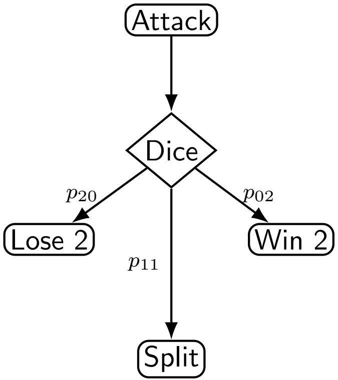
BUT! Dice make the game stochastic.



Stochastic game trees (chance nodes)

- Player chooses an action
- **Chance** (dice) chooses an outcome
- Branching without player agency

What are the actual chance of a given turnout?



Combat is a small game of its own

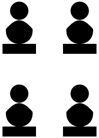
During combat, the number of dice rolled is determined by the numbers of attacking and defending units.

Scenario	Attacking units a	Defending units d	Dice rolled (A vs D)
1	$a = 1$	$d = 1$	1 vs 1
2	$a = 2$	$d = 1$	2 vs 1
3	$a \geq 3$	$d = 1$	3 vs 1
4	$a = 1$	$d \geq 2$	1 vs 2
5	$a = 2$	$d \geq 2$	2 vs 2
6	$a \geq 3$	$d \geq 2$	3 vs 2

These six scenarios fully describe all possible Risk battles.

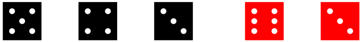
A battle, step by step (Tan's example)

Attacker

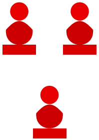


Turn 1: $(a, d) = (4, 3)$

Dice: 3 vs 2



Defender

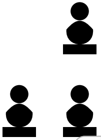


Outcome: each loses 1 $\Rightarrow$ losses (A,D)=(1,1)

Turn 1

A battle, step by step (Tan’s example)

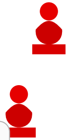
Attacker



Turn 2: $(a, d) = (3, 2)$

Dice: 3 vs 2

Defender



(Turn 2 has two ties: defender wins ties)

Outcome: attacker loses 2 (ties to defender) $\Rightarrow$ losses (A,D)=(2,0)

Turn 2

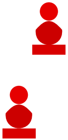
A battle, step by step (Tan's example)

Attacker

Turn 3: $(a, d) = (1, 2)$

Defender

Dice: 1 vs 2



Outcome: defender loses 1 $\Rightarrow$ losses (A,D)=(0,1)

Turn 3

A battle, step by step (Tan's example)

Attacker

Turn 4: $(a, d) = (1, 1)$

Defender

Dice: 1 vs 1

Outcome: attacker loses 1 $\Rightarrow$ losses (A,D)=(1,0)

Turn 4

A battle, step by step (Tan's example)

Attacker

Turn 5: $(a, d) = (0, 1)$

No dice (cannot attack)

Defender

Battle ends: attacker cannot continue

Turn 5 / end

Combat as a state space

We represent a battle turn by the state (a, d):

- a = attacking armies
- d = defending armies

One roll (in the 3v2 case) does one of:

$$(a, d) \to \{(a-2, d), \quad (a-1, d-1), \quad (a, d-2)\}$$

One roll in any other case:

$$(a, d) \to \{(a-1, d), \quad (a, d-1)\}$$

What are the probabilities of these transitions?

Enter: Markov chains

A Markov chain is a stochastic process with a “memoryless” property:

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

Why it fits Risk combat perfectly:

- the next battle state depends only on the current (a, d) in step n,

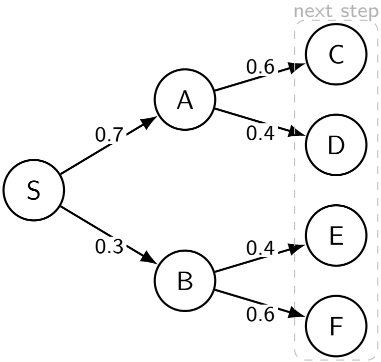
$$P(X_{n+1} = (a_{n+1}, d_{n+1}) \mid X_n = (a_n, d_n))$$

- not on how we got there.

Named after Andrey Markov (1856-1922), who studied modeling the chains of dependent events with this principle.

A toy Markov chain example

- A **Markov chain** transitions between states.
- Transition probabilities depend only on the current state
- Outgoing probabilities from a state sum to 1.



Tan's observations and transition probabilities (1997)

- When the attacker and defender start with the same number of armies, the probability that the attacker wins is **below 50%**. This is due to the fact that **ties go to the defender**.
- When the attacker has **twice as many armies as the defender**, the probability that the attacker wins the battle **exceeds 80%**.

Tan's observations and transition probabilities (1997)

Consider the combat state (a, d) with

$$a \geq 3, \qquad d \geq 2,$$

Let $X_n = (a, d)$ denote the state after the n-th roll.

The transition probabilities to the possible next states are:

$$\Pr\big[X_{n+1} = (a, d-2) \mid X_n = (a, d)\big] = \Pr\big[Y^{(1)} > Z^{(1)}\big]\ \Pr\big[Y^{(2)} > Z^{(2)}\big],$$

$$\Pr\big[X_{n+1} = (a-2, d) \mid X_n = (a, d)\big] = \Pr\big[Y^{(1)} \leq Z^{(1)}\big]\ \Pr\big[Y^{(2)} \leq Z^{(2)}\big],$$

$$\Pr\big[X_{n+1} = (a-1, d-1) \mid X_n = (a, d)\big] = 1 - \Pr\big[X_{n+1} = (a, d-2) \mid X_n = (a, d)\big] - \Pr\big[X_{n+1} = (a-2, d) \mid X_n = (a, d)\big].$$

- $Y^{(1)}, Y^{(2)}$ denote the attacker's two highest dice, and $Z^{(1)}, Z^{(2)}$ the defender's two highest dice.
- $Y^{(1)}$ and $Z^{(1)}$, as well as $Y^{(2)}$ and $Z^{(2)}$, are independent of each other

Osborne's correction (2003): dependence of ordered dice

Considers Tan's 3 vs 2 dice case:

$$\begin{aligned}\pi_{322} &= \Pr\left[Y^{(1)} > Z^{(1)},\ Y^{(2)} > Z^{(2)}\right] \\ &= \Pr\left[Y^{(1)} > Z^{(1)}\right] \ = (0.471)(0.551) = 0.259.,\end{aligned}$$

Joint distribution:

$$\begin{aligned}\pi_{322} &= \Pr\left[Y^{(1)} > Z^{(1)},\ Y^{(2)} > Z^{(2)}\right] \\ &= \sum_{z_1=1}^{5} \sum_{z_2=1}^{z_1} \sum_{y_1=z_1+1}^{6} \sum_{y_2=z_2+1}^{y_1} \Pr[Y^{(1)} = y_1, Y^{(2)} = y_2] \Pr[Z^{(1)} = z_1, Z^{(2)} = z_2] \\ &= \frac{2890}{7776} \approx 0.372.\end{aligned}$$

Key point: while individual dice are independent, the *ordered* dice $(Y^{(1)}, Y^{(2)})$ and $(Z^{(1)}, Z^{(2)})$ are *not*.

Transition probability matrix (Osborne, 2003)

i	j	Event	Symbol	Probability	Tan's value
1	1	Defender loses 1	π_{111}	$15/36 = 0.417$	0.417
1	1	Attacker loses 1	π_{110}	$21/36 = 0.583$	0.583
1	2	Defender loses 1	π_{121}	$55/216 = 0.255$	0.254
1	2	Attacker loses 1	π_{120}	$161/216 = 0.745$	0.746
2	1	Defender loses 1	π_{211}	$125/216 = 0.579$	0.578
2	1	Attacker loses 1	π_{210}	$91/216 = 0.421$	0.422
2	2	Defender loses 2	π_{222}	$295/1296 = 0.228$	0.152
2	2	Each loses 1	π_{221}	$420/1296 = 0.324$	0.475
2	2	Attacker loses 2	π_{220}	$581/1296 = 0.448$	0.373
3	1	Defender loses 1	π_{311}	$855/1296 = 0.660$	0.659
3	1	Attacker loses 1	π_{310}	$441/1296 = 0.340$	0.341
3	2	Defender loses 2	π_{322}	$2890/7776 = 0.372$	0.259
3	2	Each loses 1	π_{321}	$2611/7776 = 0.336$	0.504
3	2	Attacker loses 2	π_{320}	$2275/7776 = 0.293$	0.237

π_{ijk} is the probability that, when attacker rolls i dice and defender rolls j dice, the defender loses k units (so attacker loses $j-k$ units).

The battle as a Markov chain: start and end states

- A Risk battle can be modeled as a Markov chain on states

$$X_n = (a, d),$$

where a is the number of **attacking units** still able to fight and d is the number of **defending units** remaining.

- The battle starts at the initial state

$$X_0 = (A, D),$$

where A and D are the initial attacking and defending unit counts.

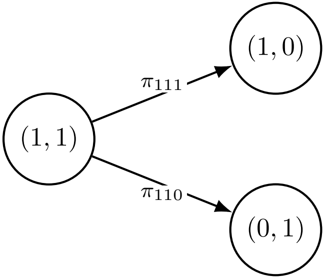
- The chain ends when it hits an **absorbing boundary**:
  - **Attacker wins:** $(a, 0)$ for some $A \geq a \geq 1$.
  - **Defender holds:** $(0, d)$ for some $D \geq d \geq 1$.

Building the battle Markov chain: the base case $(1,1)$

From the transition table (case $i = j = 1$):

- $\pi_{111} = \Pr(\text{defender loses 1}) = 15/36$
- $\pi_{110} = \Pr(\text{attacker loses 1}) = 21/36$

So the state $(1,1)$ transitions directly to an absorbing boundary.



This is the smallest nontrivial Markov chain: one transient state, two absorbing outcomes.

Building the chain: from $(2,1)$ to $(1,1)$ and beyond

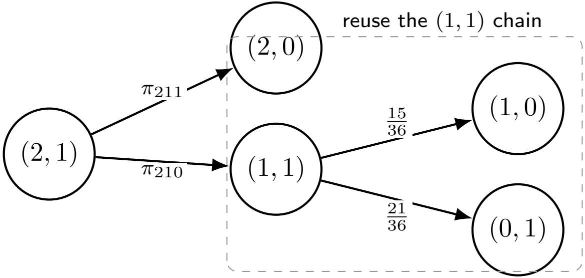
From the transition table (case $i = 2,\ j = 1$):

- $\pi_{211} = \Pr(\text{defender loses 1}) = 125/216$
- $\pi_{210} = \Pr(\text{attacker loses 1}) = 91/216$

So:

$$(2,1) \to (2,0) \text{ or } (1,1).$$

But $(1,1)$ is the starting state of our previous Markov chain!



We build larger battles by *reusing* smaller ones: this is exactly the Markov-chain recursion.

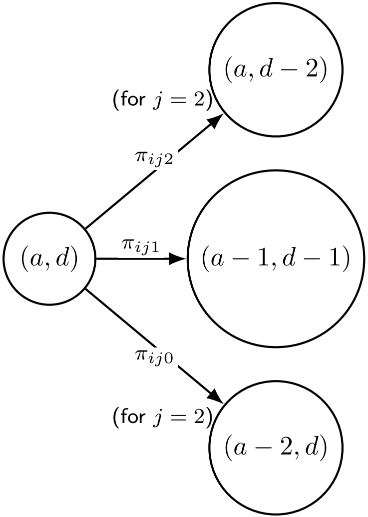
The general rule

At any state (a, d), the dice scenario (i, j) is determined by the rules:

$$i = \min(3, a), \qquad j = \min(2, d).$$

Three possible outcomes from the transition table:

- Defender loses j units: $(a, d) \to (a,\ d-2)$ with probability $\pi_{ij\,j}$
- Each loses 1 (only when $j = 2$): $(a, d) \to (a-1,\ d-1)$ with probability $\pi_{ij\,1}$
- Attacker loses j units: $(a, d) \to (a-j,\ d)$ with probability $\pi_{ij\,0}$

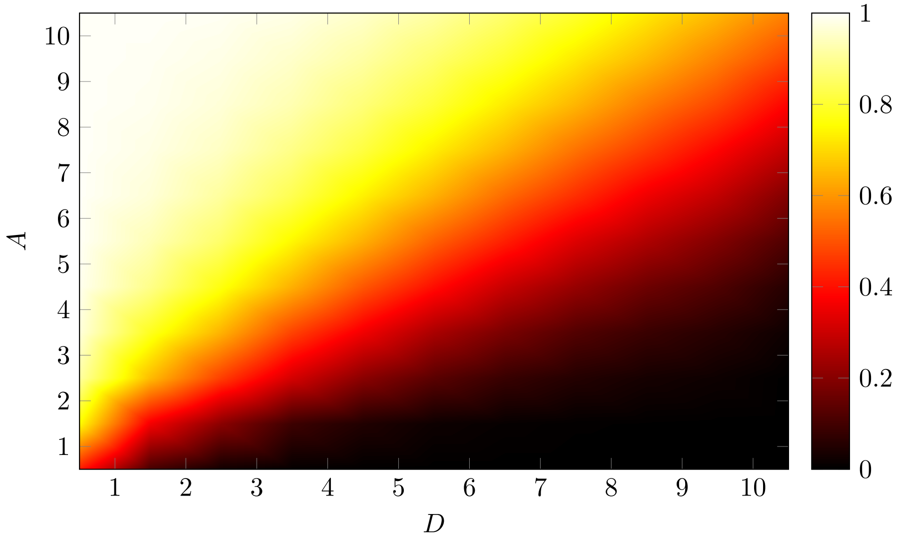


If the transition probability matrix is known, the entire battle Markov chain can be constructed.

Probability that the attacker wins

D \ A	1	2	3	4	5	6	7	8	9	10
1	0.417	0.106	0.027	0.007	0.002	0.000	0.000	0.000	0.000	0.000
2	0.754	0.363	0.206	0.091	0.049	0.021	0.011	0.005	0.003	0.001
3	0.916	0.656	0.470	0.315	0.206	0.134	0.084	0.054	0.033	0.021
4	0.972	0.785	0.642	0.477	0.359	0.253	0.181	0.123	0.086	0.057
5	0.990	0.890	0.769	0.638	0.506	0.397	0.297	0.224	0.162	0.118
6	0.997	0.934	0.857	0.745	0.638	0.521	0.423	0.329	0.258	0.193
7	0.999	0.967	0.910	0.834	0.736	0.640	0.536	0.446	0.357	0.287
8	1.000	0.980	0.947	0.888	0.818	0.730	0.643	0.547	0.464	0.380
9	1.000	0.990	0.967	0.930	0.873	0.808	0.726	0.646	0.558	0.480
10	1.000	0.994	0.981	0.954	0.916	0.861	0.800	0.724	0.650	0.568

Probability that the attacker wins



Osborne's conclusions

- The chances of winning a battle are **considerably more favorable for the attacker** than originally suggested by Tan.
- A **more aggressive attacking strategy** is often justified.
- When $A = D$, the attacker has a **higher than 50%** to conquer the territory, provided that $A \geq 5$.
- Also, when $A = D$ and A is not small, the attacker **suffers fewer losses on average** than the defender.

Zooming back out

- Battles are local Markov chains (solvable exactly).
- But the full game is:
  - huge,
  - multi-player,
  - strategic and psychological.
- So a strong player:
  - exploits the topology,
  - uses probability to their advantage,
  - and excels at politics.

Conclusion

- This talk began as a trip down memory lane, revisiting an old 'frenemy' from years ago.
- Revealed to be a deep and rich topic.
- The problem space is supported by a substantial body of scientific literature.
- Opportunities for developing and evaluating gaming agents and strategic decision-making models.
- Active and passionate players.

Thanks!

Thank you for your attention

References

Scientific literature

- Tan, B. (1997). *Markov chains and the RISK board game*. *Mathematics Magazine*, 70(5), 349–357.
- Osborne, J. A. (2003). *Markov Chains for the RISK Board Game Revisited*. *Mathematics Magazine*, 76(2), 129–135.

Image and media sources

- Trikorder Blog. *10 társasjáték a gyerekkorból*. https://trikorder.blog.hu/2016/01/19/10_tarsasjatek_a_gyerekkorbol
- Gremlin.hu. *Rizikó (1957)*. https://gremlin.hu/jatekok/riziko-1957
- Saseskos.hu. *Ki nevet a végén – fa dobozos verzió*. https://www.saseskos.hu/ki_nevet_a_vegen_osszehajthato_fa_dobozos_verzio_4961
- BoardGameGeek. *Tikal image*. https://boardgamegeek.com/image/178078/tikal
- BoardGameGeek. *Catan image*. https://boardgamegeek.com/image/1402774/catan

References (continued)

Image and media sources (continued)

- BoardGameGeek. *Risk images.*
  https://boardgamegeek.com/image/452643/risk
  https://boardgamegeek.com/image/1930033/risk
- Wikipedia contributors. *Risk (game).*
  https://en.wikipedia.org/wiki/Risk_(game)
- Wikipedia contributors. *Chessboard.*
  https://en.wikipedia.org/wiki/Chessboard

Recommended resources

- Hasbro. *Risk official rulebook.*
  https://www.hasbro.com/common/instruct/risk.pdf
- YouTube (Numberphile). *The Game of Risk.*
  https://www.youtube.com/watch?v=RdooKXXcWWc
- YouTube playlist (Brian Veitch). *Probabilities of a Risk battle* .
  https://www.youtube.com/playlist?list=PL9kH3mCPsggcdwgZhhMm6iZ0VdTiBGxEZ
- YouTube (Veritasium). *The Strange Math That Predicts (Almost) Everything.*
  https://www.youtube.com/watch?v=KZeIEiBrT_w