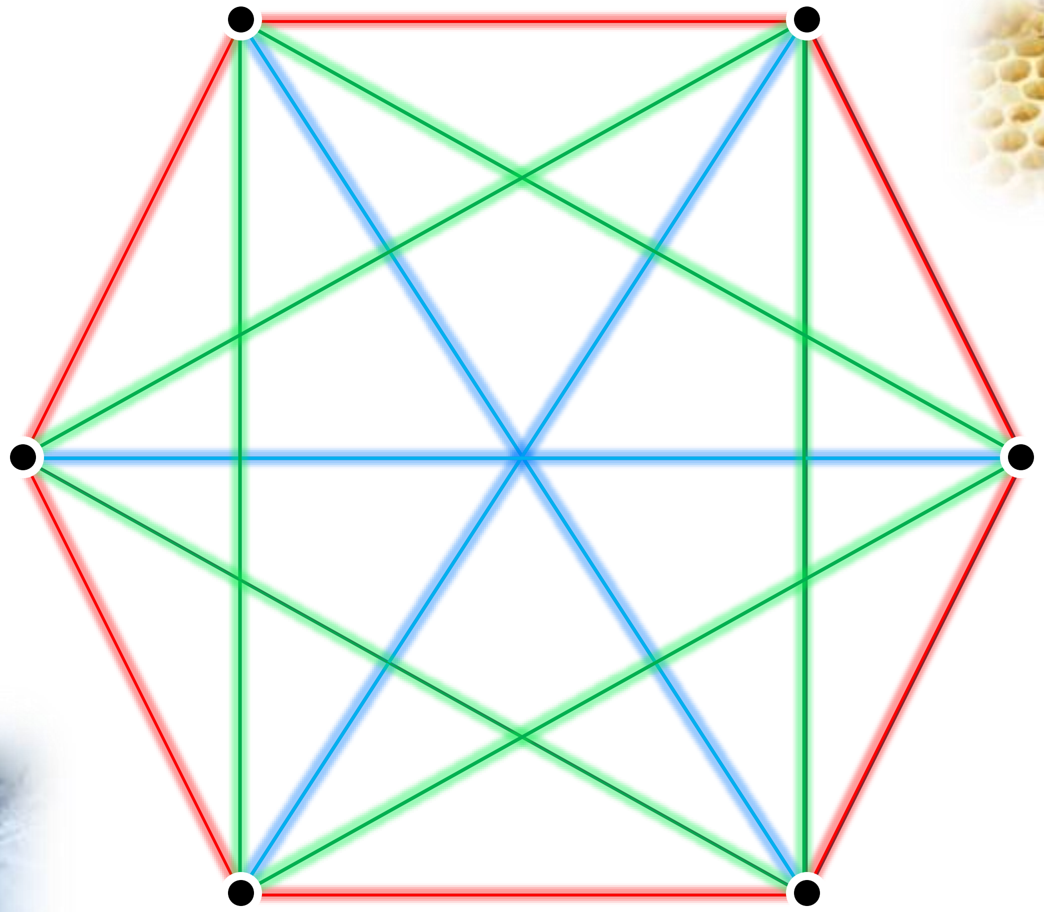
Maths Has Colours

*Beyond the Grayscale: How **Association Schemes** Reveal the True Colours of Symmetry*



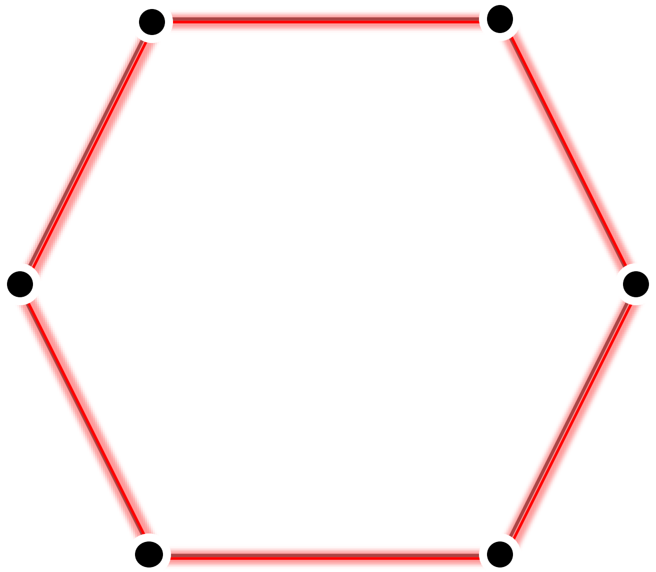
Giusy Monzillo

UP Famnit

April 22nd, 2026

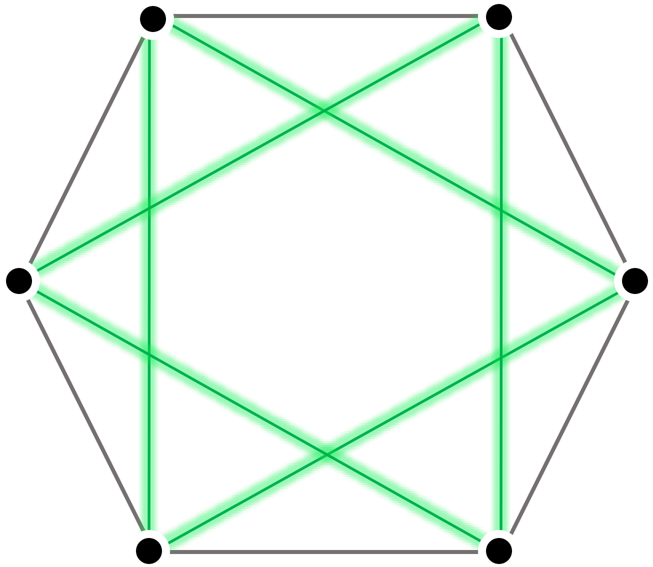
Symmetry as Distance: Decoding the "Social Rules" of the Hexagon

The Inner Circle



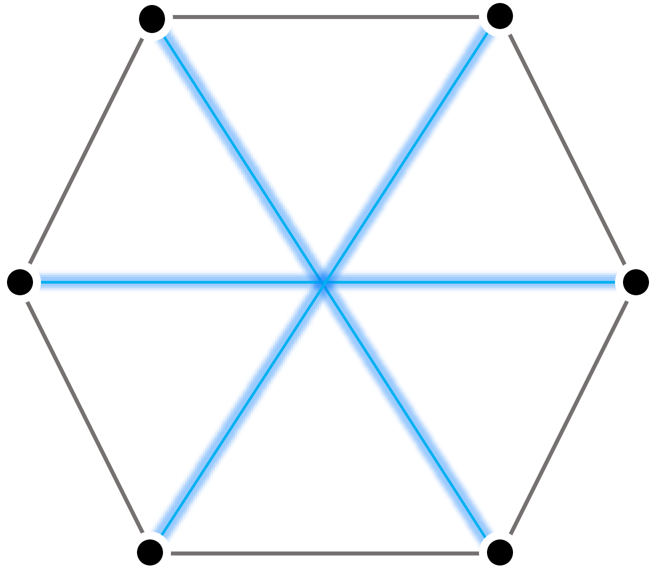
Your immediate neighbours: Your **Best Friends**
(Distance 1)

Friends of Friends



You share a common connection: Your **Acquaintances**
(Distance 2)

The Farthest

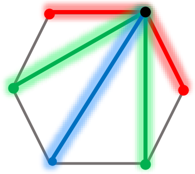


You are at the maximum distance: The **Total Strangers**
(Distance 3)

💡 *Mathematical Fairness*

In an AS, every point is **identical**. No matter which vertex you pick, you will find:

Exactly **2 Red** connections
Exactly **2 Green** connections
Exactly **1 Blue** connection.



- The colours **partition** the edges of the complete graph K_6 into three classes.
- The colour classes form a **3-class AS**.
- The cycle graph C_6 is **Distance-Regular**.

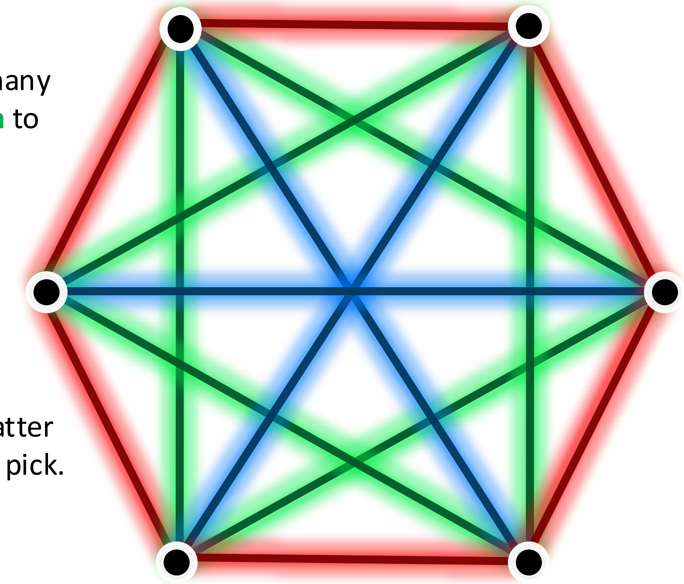
The Rules of Interaction: Calculating "Common Ground" in a Symmetric World

Ex. 1: The "Stranger" Mystery

Q: If we are **Blue**, how many are **Red** to me but **Green** to you?

$$p^{B}_{RG} = p^{B}_{GR} = 2$$

A: In an AS, it doesn't matter *which* two strangers you pick. The answer is **always 2**.

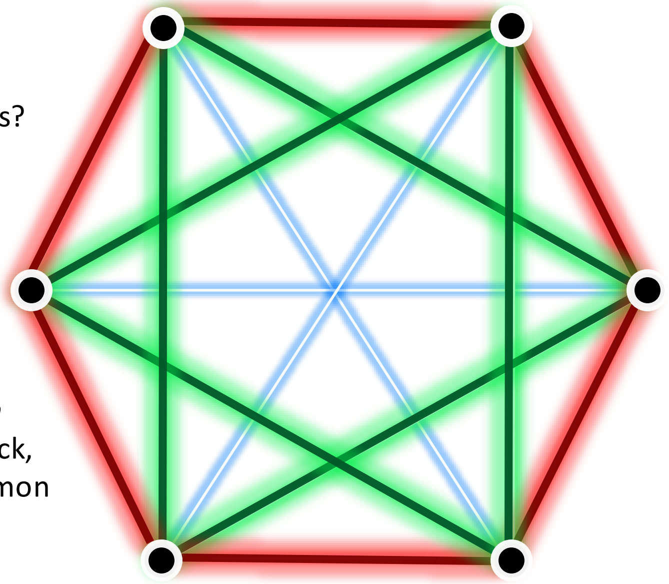


Ex. 2: The "Mutual Friend" Mystery

Q: If we are **Green**, how many are **Red** to both of us?

$$p^{G}_{RR} = 1$$

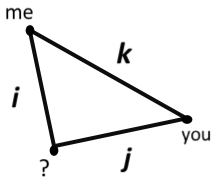
A: Again, no matter *which* two acquaintances you pick, they share **exactly 1** common friend.



💡 *General Rule*

Q: If we are in relationship $\boldsymbol{k}$, how many people are in relationship $\boldsymbol{i}$ with me and $\boldsymbol{j}$ with you?

A: In AS, this number must be **constant**.

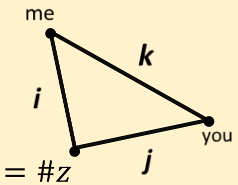


The Intersection Numbers

If our relationship is colour $\boldsymbol{k}$, the symbol $\boldsymbol{p^{k}_{ij}}$ is the number of $\boldsymbol{z}$ such that:

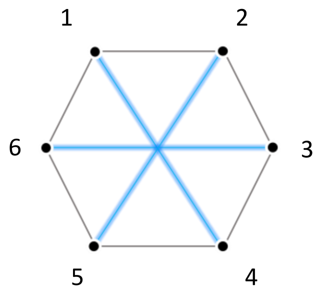
- The relationship between **me and $\boldsymbol{z}$** is colour $\boldsymbol{i}$.
- The relationship between **you and $\boldsymbol{z}$** is colour $\boldsymbol{j}$.

The Magic: This number depends **only on $\boldsymbol{i, j, k}$.** $p^{k}_{ij} = \#z$

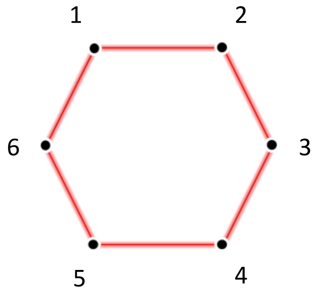


From Pictures to Matrices:
Turning the Hexagon into a "Math Machine"

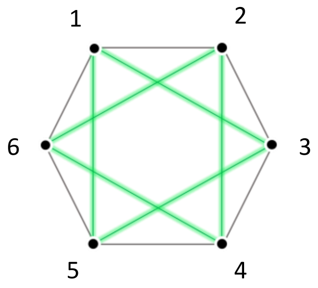
$$A_B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



$$A_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

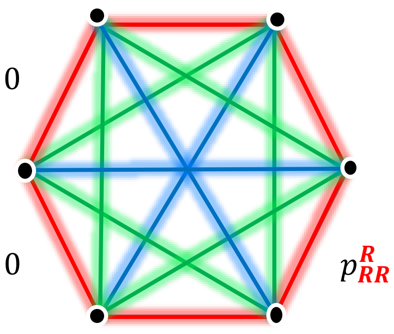


💡 *Matrix Secret*

$$A_R A_G = p_{RG}^{R} A_R + p_{RG}^{B} A_B = A_R + 2A_B$$

$$A_R{}^2 = 2\, I_6 + p_{RR}^{G} A_G = 2I_6 + A_G$$

$p_{RG}^{G} = 0$



$p_{RR}^{B} = 0$

$p_{RR}^{R} = 0$

The Algebra of the Scheme

In an AS, these matrices have "superpowers":

Property 1: Together with the identity matrix I_6, they add up to the "all-ones" matrix.

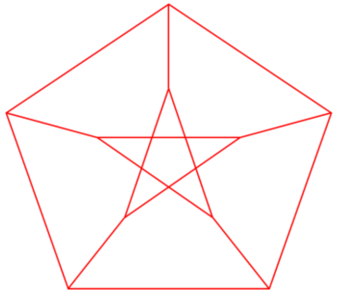
Property 2: They are symmetric.

Property 3: They commute, and $\boldsymbol{A_i A_j = \sum p_{ij}^{k} A_k}$

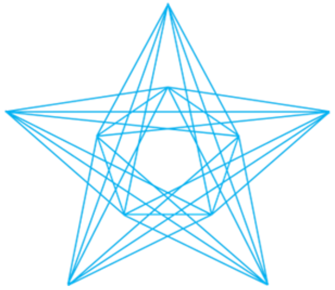
Beyond the Hexagon: The Petersen Graph

The (Main) Difference

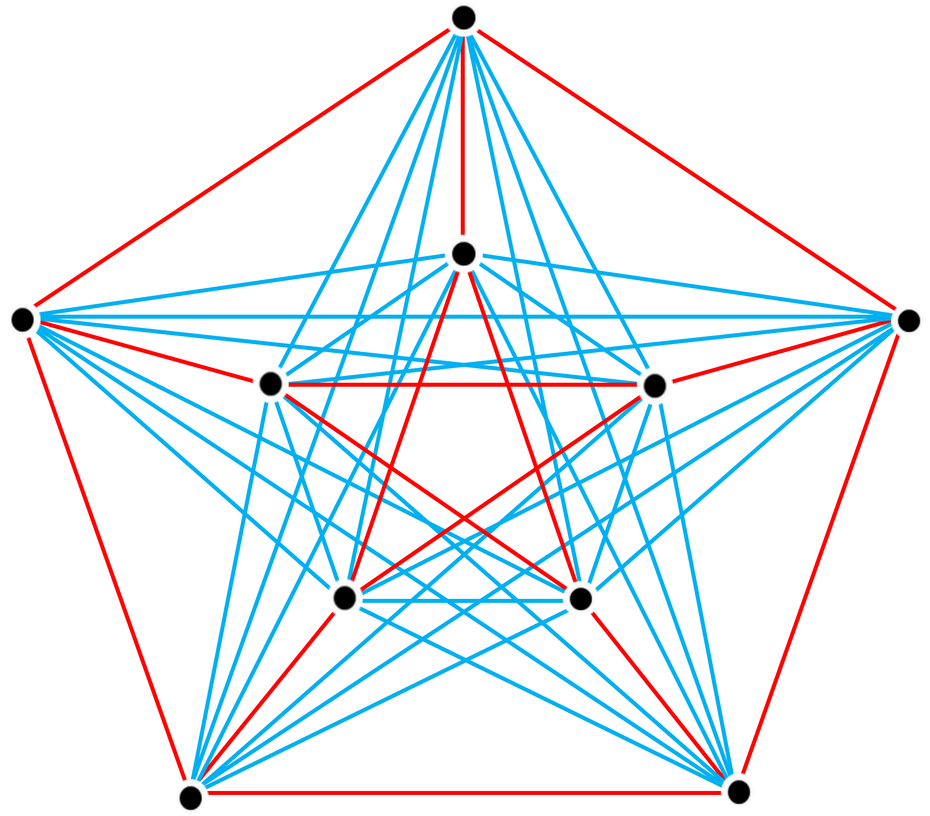
Unlike the Hexagon, the **Petersen graph** requires only **2 colours** (Red and Blue) because the maximum distance between any two points is **2 steps**.



Distance-1 graph



Distance-2 graph



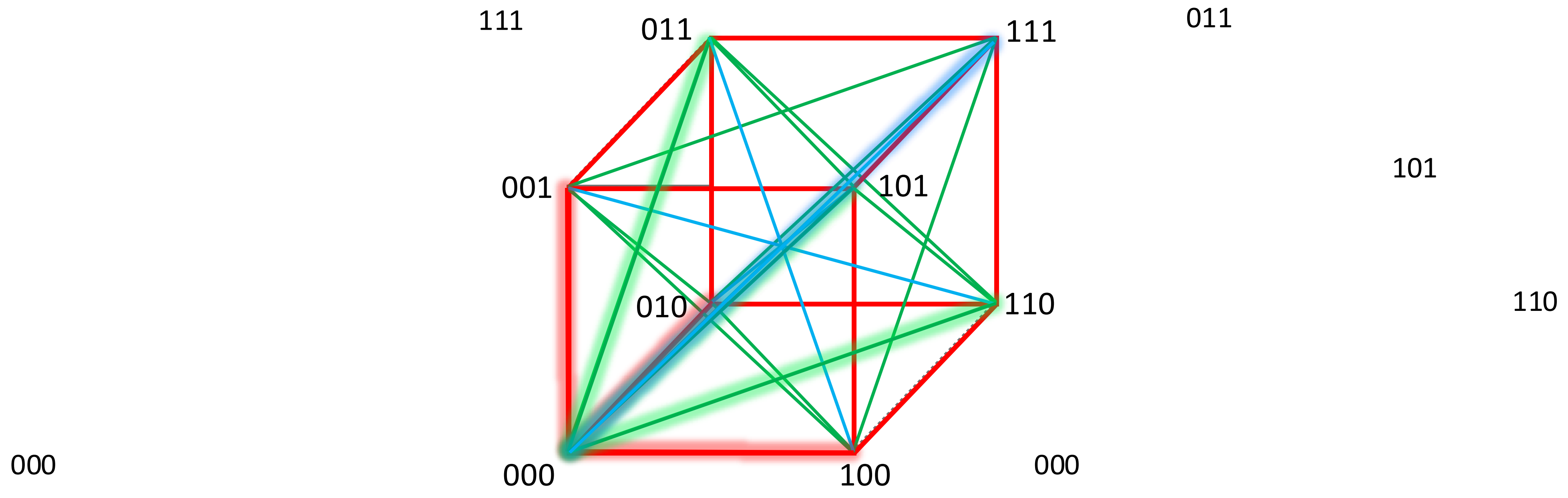
Property	Hexagon	Petersen
Vertices	6	10
Classes	R,G,B	R,B
Red Connections	2	3
Green Connections	2	0
Blue Connections	1	6

💡 *Special Parameters* $(10,3,0,1)$

This is the **Strongly Regular Graph** $\mathrm{SRG}(10,3,0,1)$, where

- **10**: People in the room.
- **3**: Friends each person has (Red Connections).
- **0**: Friends shared by two people who are *already* friends ("No-Triangle" rule: $p^{R}_{RR} = 0$).
- **1**: Friend shared by two people who are *not* friends ($p^{B}_{RR} = 1$).

The Hamming Scheme: The Language of Bits



Any two words differ by **3 bits**
(**Body Diagonals** – Distance 3)

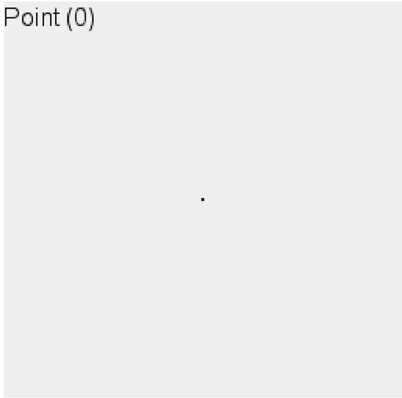
Any two words differ by **1 bit**
(**Edges** – Distance 1)

Any two words differ by **2 bits**
(**Face Diagonals** – Distance 2)

✨ The **3D Cube** is the *perfect* example of a **Hamming Scheme**. It is how computers see the world: **binary codes** and **flipped bits**.

This is how **Error-Correcting Codes** work. If NASA sends 000 and it arrives as 001, the computer sees Red and knows there was a 1-bit error.

The Hamming Scheme: Scaling the Cube



$H(n,2)$ Dimension Transitions

The **Hamming Scheme** is the "Gold Standard" for **Coding Theory.**

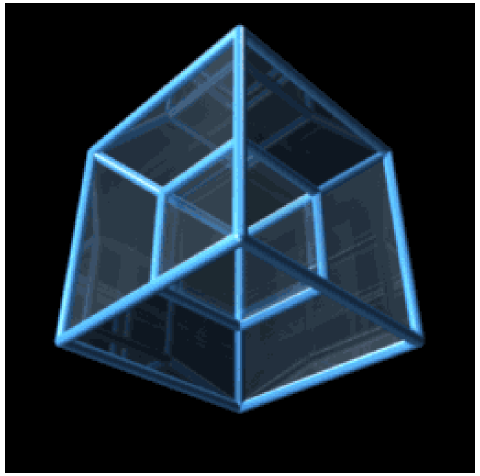
The World: All possible n-tuples (strings of length n) using an alphabet of size q.

The Colour Classes: Two "words" form a colour-i edge if they differ in exactly i positions – **Hamming Distance.**

Notation:

- $H(n,2)$ is the standard n-dimensional **Hypercube** (for $n = 3$, we have our **3D Cube**).
- $H(n,q)$ describes more complex data (like DNA sequences for $q = 4$).

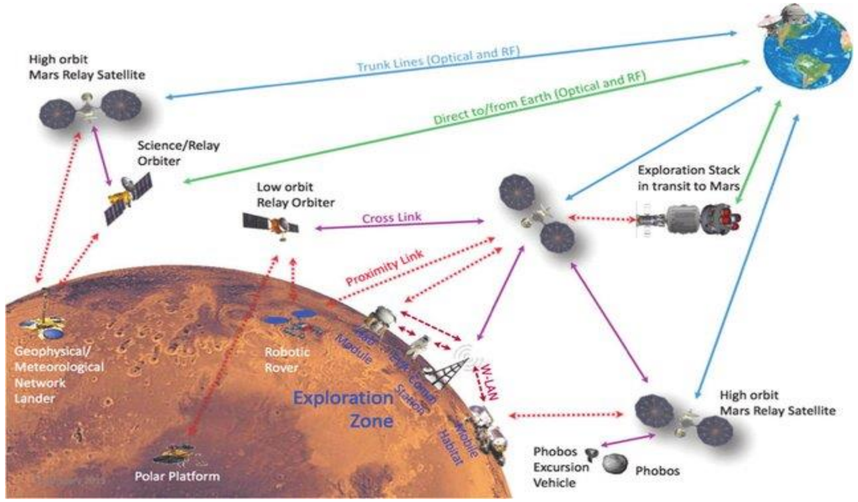

La Grande Arche de la Défense (Paris, France)


3D projection of a tesseract - 4D Cube

✨ #World = q^n
#Colours = n

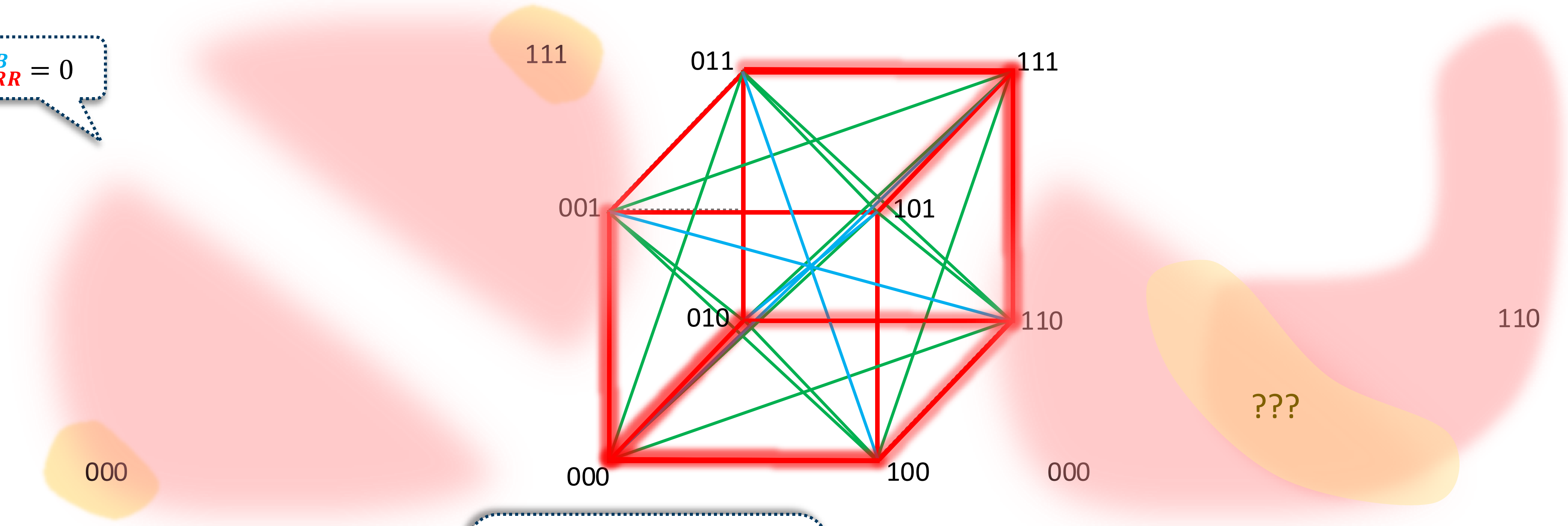
✨ We need a **4-colour palette** for a 4D Cube.

The maths NASA uses to talk to Mars is the same maths Nature uses to store the blueprint of life. An AS is not just an abstract concept; it is the universal language of **stable information**, whether that info is stored in a **silicon chip** or a **carbon cell**.



The Error-Correcting Power of the Hamming Scheme

$p_{RR}^{B} = 0$



$p_{RR}^{G} = 2$

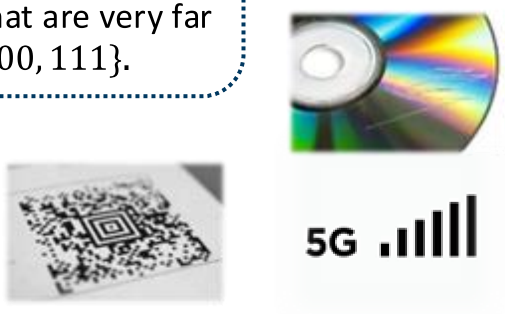
No Overlap - Total Safety:
If a bit flips, we know exactly how to fix it.

💡 If NASA wants to send a message safely, it picks a **Code** consisting of only **Codewords** that are very far apart, e.g., $\{000, 111\}$.

Overlap - Ambiguity:
If a message falls in the intersection, we do not know how to fix it.

✨ If a message falls in the Red Bubble around 000, it **must** be 000 with an **error**: the computer corrects it instantly.
The same applies to 111.
This is the **Error-Correcting Power** of our AS.

NASA's DSS-43
(Tidbinbilla, Australia)



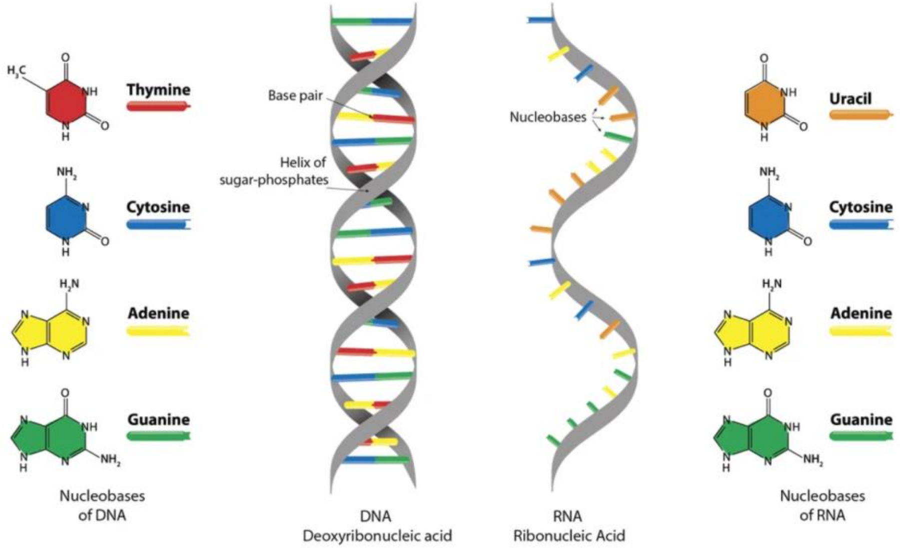
Hamming Scheme:
Our Superhero against Digital Noise

The Golden Rule
To correct t errors, the **Minimum Distance** d must satisfy:

$$d \geq 2t + 1.$$

In our case, for **1 error** ($t = 1$), we require $d = 3$.
This is why Blue is our salvation, while Green fails!

The Biological Hamming Scheme



✨ #World = $4^3 = 64$
#Colours = 3

Life operates in the $\boldsymbol{H}(\mathbf{3},\mathbf{4})$ Hamming Space.

The Alphabet: DNA uses **4 letters** (bases), i.e., **A**, **C**, **G**, and **T**.

The Words: These letters are grouped into **3-letter words**, i.e., the **Codons**.

Each word is a recipe for an **Amino Acid**, the building blocks of our body. There are **20** of them!

💡 *Why did nature choose 3 letters?*

- $n = 2 \rightarrow 4^2 = 16$ (insufficient for the 20 aminos)
- $n = 3 \rightarrow 4^3 = 64$: **Redundancy Buffer**.

This creates the "Extra Room" needed for the **Biological Error-Correction System** (Repair Enzymes) to identify and fix mutations.

The Redundancy Safety Net
The mapping from 64 codons to 20 aminos is "many-to-one".
This isn't random; it's a **strategic mathematical layout**!
Nature concentrates most of its redundancy on **the 3rd position of the codon,** as it is the most prone to "noise" (UV rays, chemicals, heat, etc), e.g.,

$$\{GGA, GGC, GGG, GGT\} \mapsto Glycine$$

Mathematical Impact: Most Distance-1 errors at the 3rd position are "silent".

✨ If GGA changes into GGC (or GGG, or GGT), the result will stay the same, i.e., Glycine. This is a **Silent Mutation**.

Gly

GGA

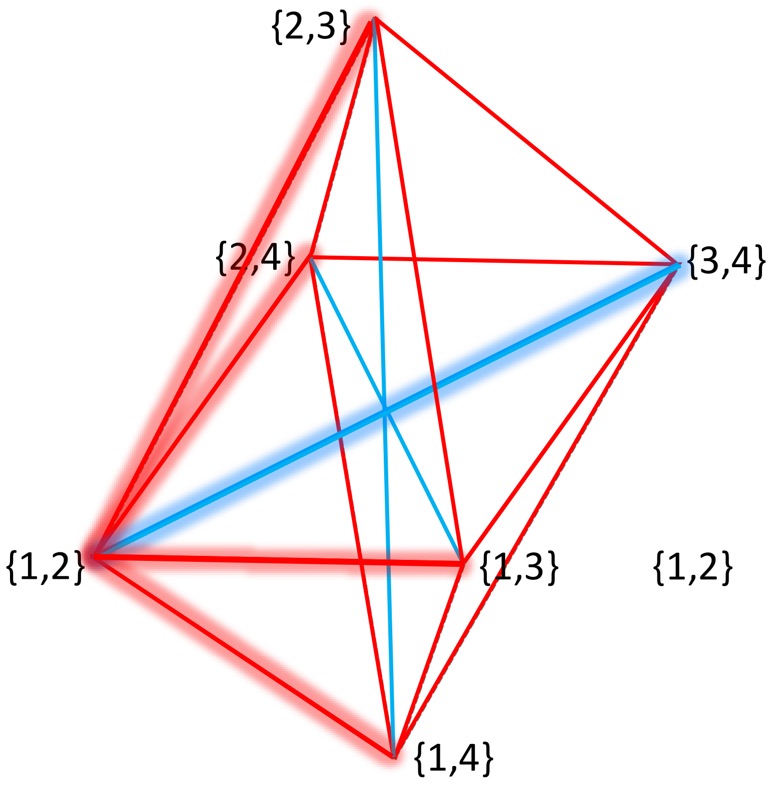
The Johnson Scheme: The Symmetry Behind the Curtain

Questions

- How many groups of k people can we form out of a population of v people?
- How do these k-groups "interact"?

Once again, the answer sits in the ASs. In case of pairs ($k = 2$), we need only **2 colours** (Red and Blue) because any two pairs differ in at most **2 people**.

Colours depend on the **Intersections**.



{3,4}

{1,2}

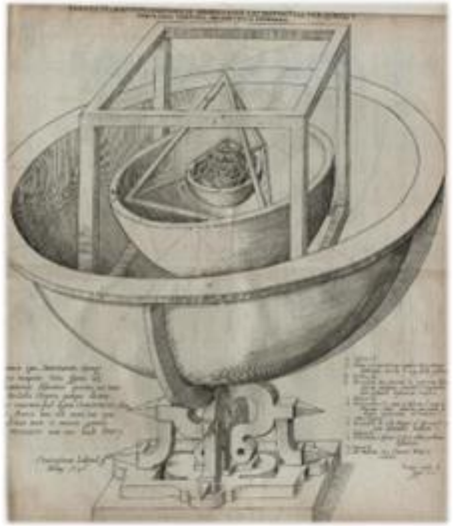
Any two sets intersect in **1 element**
(**Edges** – Distance 1)

Any two sets intersect in **0 elements**
(**Diagonals** – Distance 2)

✨ The **Octahedron** with its diagonals forms the most *intuitive* (non-trivial) example of a **Johnson Scheme**.
It reveals the **hidden regularity** in how we pick and choose groups.

✨ Johnson Schemes are strictly related to the way Nature builds and organizes its **Perfect Shapes** – **Platonic Solids**.

The Johnson Scheme: Beyond the Octahedron



Kepler's Platonic Solid Model
of the Solar System (1596)

The **Johnson Scheme** is the "Heart" of **Design Theory.**

The World: All possible k-subsets (sets of size k) of a set of size v ($\{1,2,\ldots,v\}$).

The Colour Classes: Two sets form a colour-i edge if they share exactly $k-i$ elements.

Notation:

- $J(v,k)$ is also called the **Triangular Scheme.**
- $J(v,k) \cong J(v,v-k) \;\rightarrow\; k \leq v/2$

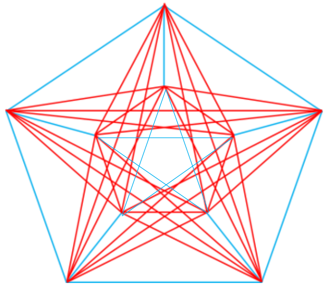
✨ #World = $\binom{v}{k}$
#Colours = k

✨ $J(4,2)$:
Octahedron

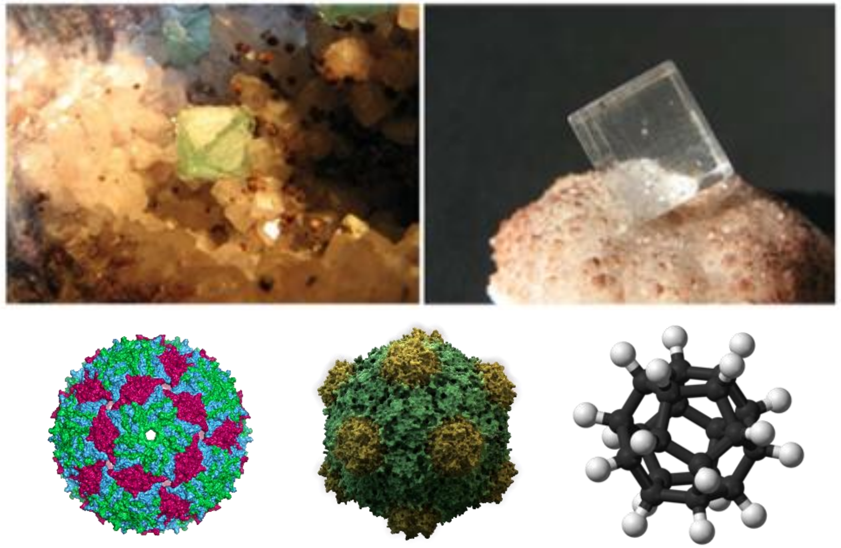
- The Red Graph of $J(v,k)$ is the **Johnson Graph** – Neighbourhood Graph.
- The Blue Graph of $J(v,k)$ is the **Kneser Graph** – Maximum-Distance Graph.

💡 *What is $J(5,2)$?*
Hint: **10** points and **2** colours!

The **Petersen Graph** is the Kneser Graph of $\boldsymbol{J(5,2)}$.



Colour Swap: The Petersen is Blue to a Johnson



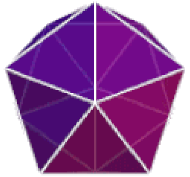
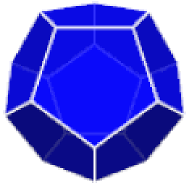
Platonic Solids in Crystals, Viruses, and Hydrocarbons

Often, the Beauty of Symmetry means Regularity, but also **Efficiency**. The ancient Greeks identified the fundamental elements of the universe with the **five Platonic Solids** – structures that both Nature and Humanity have adopted for millennia to shape our world in the most efficient and elegant ways possible.

Platonic Solids in Puzzles and Space Frames

The Association Schemes of the Platonic Solids: Not Only Johnson Schemes



	Tetrahedron	Hexahedron (Cube)	Octahedron	Icosahedron	Dodecahedron
Vertices	4	8	6	12	20
Faces	4	6	8	20	12
Edges	6	12	12	30	30
Colours	1	3	2	3	5
What AS?	$\boldsymbol{J(4,1) - K_4}$	$\boldsymbol{H(3,2)}$	$\boldsymbol{J(4,2)}$	**Icosahedral Scheme**	**Dodecahedral Scheme**

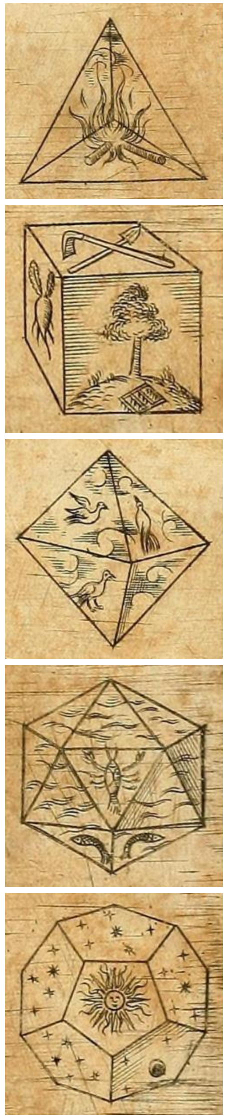
💡 If we interchange the faces with the vertices, we obtain the **Dual** of a Platonic Solid:

- The Tetrahedron is **Self-Dual**.
- The Cube and the Octahedron are a **Dual Pair**.
- The Icosahedron and the Dodecahedron, too.

💡 *Crushing Platonic Solids*

- If you "squash" the vertices of the Cube onto a plane perpendicular to any Body Diagonal, you see the **Hexagon**.
- If you "merge" every Antipodal Pair of the Icosahedron, you see $\boldsymbol{K_6 - J(6,1)}$.
- If you apply the same to the Dodecahedron, you see the **Petersen** $\boldsymbol{- J(5,2)}$.

Through the Lenses of Platonic Solids



The five Elements as Platonic Solids

A **Platonic Solid** is a 3D (convex) **polyhedron** where every face is an **identical regular polygon** and the same number of faces meet at each vertex.

From Ancient Philosophy to Modern Science

- *Ancient Roots*: **Plato** (360 BC) associated them with the fundamental elements: **Fire** (Tetrahedron), **Earth** (Cube), **Air** (Octahedron), **Water** (Icosahedron), and the **Cosmos** (Dodecahedron).

- *The Scientific Revolution*: **Kepler** (1596 AC) attempted to explain the **Solar System** using a nested model of these solids.

- *Today's Lens*: We now see and use them as ASs to solve complex problems in **Coding Theory**, **Crystallography**, **Molecular Biology**, **Meteorology**, and **Statics**.

Tetrahedron (Bottrop, Germany)

Solar Cube at Discovery Cube Orange County (California, USA)

Octahedron (Sigleß, Austria)

Icosahedron from Spinoza monument (Amsterdam, The Netherlands)

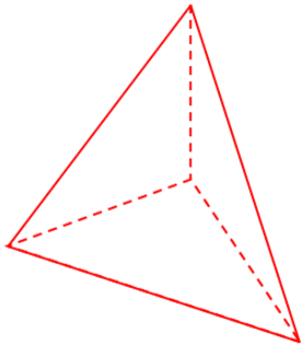
Dodecahedron (Budapest, Hungary)

✨ **Theaetetus' Theorem** (ca. 417-369 BC)
There are **precisely five Platonic solids**.

They are foundational in **gaming** for creating *fair dice* and in **puzzles** like *magic polyhedra*.

Their symmetry, structural integrity, and beauty have inspired **architects**, **artists**, and **artisans** from ancient Egypt to the present.

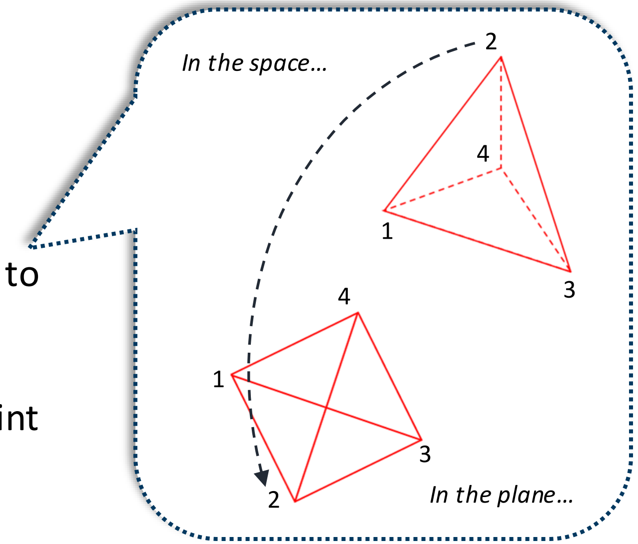
The Tetrahedron: The Simplicity of Balance



The Complete Scheme $J(4,1)$

• *The Maths*: It is the simplest AS. Every vertex is connected to every other vertex – **Complete Graph K_4**.

• *The Colour Palette*: Only **One Colour** (Distance 1) – every point is a neighbour of any other.



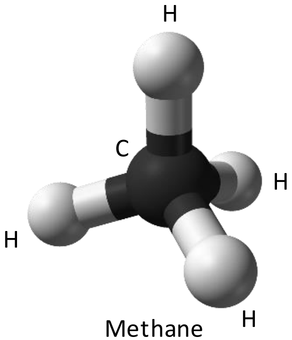
C_4H_4 is a *hypothetical* **Platonic Hydrocarbon**.

Tetrahedrite

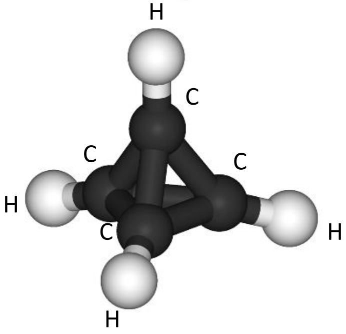
Nature's Building Block

• *Chemistry*: **Methane** (CH_4) – One **carbon** sits in the center, and **4 hydrogens** push away to the corners of a perfect tetrahedron to reach **maximum stability**.

• *Mineralogy*: **Tetrahedrite** – This mineral grows into nearly perfect, sharp-edged tetrahedra. It is an ore of **copper** and associated metals.



Methane



Tetrahedrane

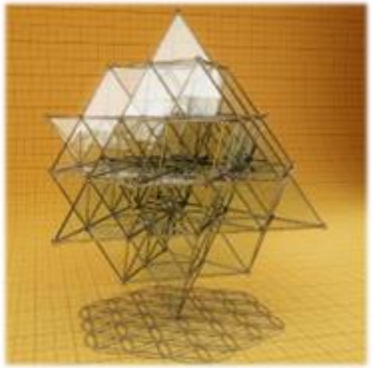
Human Engineering

• **Tetra Pak**: The original "Tetra Classic" milk carton was a Tetrahedron – minimum amount of packaging material for a stable, **stackable volume**.

• **The 64 Tetrahedron Grid**: 3D fractal structure made of 64 tetrahedra, believed to represent the fundamental **vacuum** structure of spacetime at all scales.

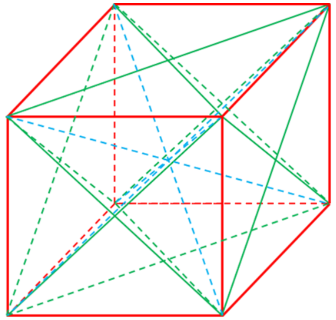
Tetra Pak milk carton advertisement, 1950

Swedish Tetra Classic packaging



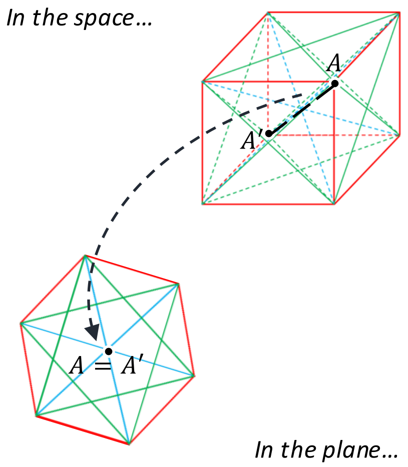
64 Tetrahedron Grid and its 2D shadow - "Flower of Life"

The Hexahedron: The Pillar of Stability

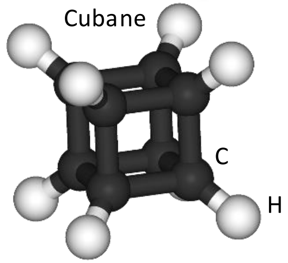


The Hamming Scheme $H(3,2)$

This is the Geometry of **Binary Codes**.



$C_8\,H_8$ is a **Platonic Hydrocarbon**.



• *The Maths*: It is the **3D Hypercube – Cube**. It arises from a **Distance-Regular Graph** (Red Cube) with maximum distance 3.

• *The Colour Palette*: **3 Colours** – Edges (Dist. 1), Face Diagonals (Dist. 2), and Body Diagonals (Dist. 3).

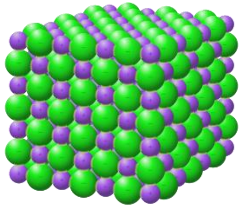
Pyrite Crystals

Nature's Grid

• *Chemistry*: **Salt** (**NaCl**) – Sodium (Na) and Chloride (Cl) ions interlock in a perfect cubic lattice. It is the most **energy-efficient** way to balance electric charges.

Salt Crystals under Microscope

• *Mineralogy*: **Pyrite** (**Fool's Gold**) – This mineral grows into cubes so perfect they look man-made. It proves that Nature "thinks" in terms of Hamming-like regularity to create **structural order**.



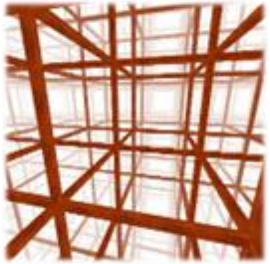
NaCl Crystal Structure

Human Engineering

• *Technology*: **5G & NASA** – The Cube is the "Superhero of Data". Digital signals use the 3 distances of the Cube to detect and fix "bit-flips" caused by **noise**.

• *Architecture*: **Cubic Honeycomb** – The Cube is the global standard for **spatial efficiency**. It is the only Platonic Solid that can **tessellate** 3D space without gaps.

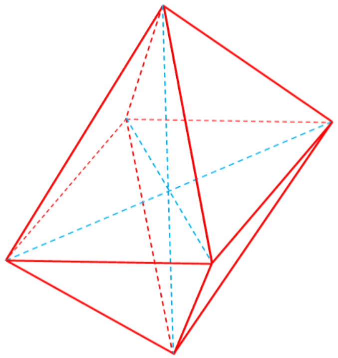
Habitat 67 (Montreal, Canada)



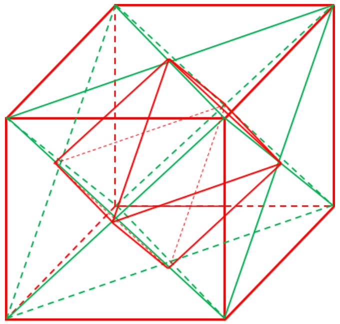
Cubic Honeycomb Grid

NASA's Voyager 1

The Octahedron: The Geometry of Strength



*It is the **Dual** of the Cube!*



The Johnson Scheme $J(4,2)$

• *The Maths*: It arises from a **Distance-Regular Graph** (Red Octahedron) with maximum distance 2 – **Strongly-Regular Graph**.

• *The Colour Palette*: **2 Colours** – Edges (Dist. 1) and Body Diagonals (Dist. 2).

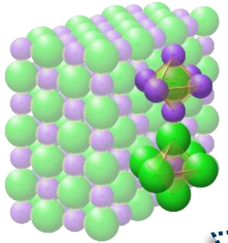
Nature's Armour

Raw Diamond (0.105ct)

• *Crystallography*: **Diamond** – Raw diamond crystals grow as perfect octahedra. The $J(4,2)$ symmetry allows carbon atoms to bond in the **most rigid** way possible.

• *Mineralogy*: **Magnetite** – This natural magnet (one of the main iron ores) also forms perfect octahedra, balancing **magnetic forces** through regular symmetry.

Magnetite Crystals

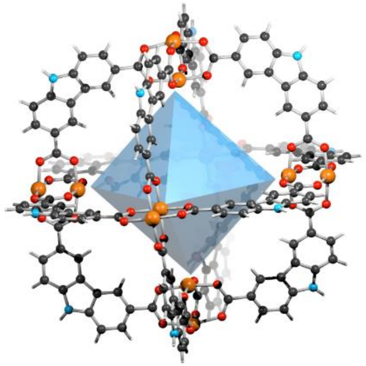


The rock-salt structure has **Octahedral Coordination**.

Human Engineering

• *Technology*: **Molecular Cages** (**MOFs**) – Nanoscopic "cages" with octahedral symmetry to act as a **molecular sieve**. The $J(4,2)$ geometry creates the perfect "trap" to catch CO_2 in the air.

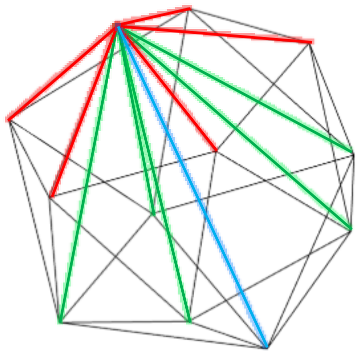
• *Architecture*: **Octet Truss** – This **Space Frame**, interlocking octahedra and tetrahedra, forms an **inherently rigid** grid, used for **stadium roofs**, **space stations**, **airports**, etc.

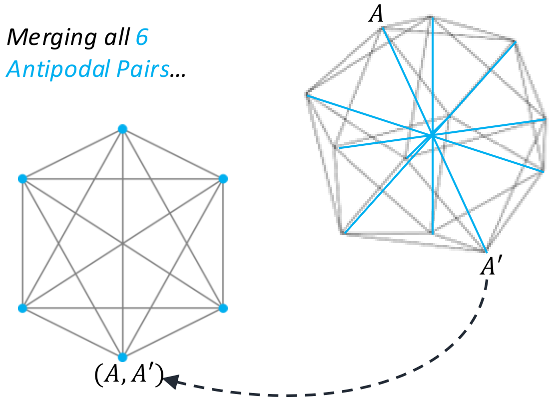


Metal-Organic Framework

INTAÇ's Space Roof

The Icosahedron: The Efficiency of Protection

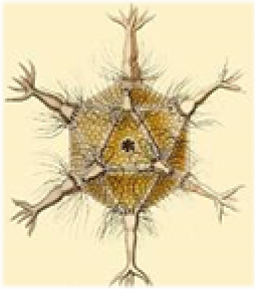




📐 The Icosahedral Scheme

• *The Maths*: The Icosahedron is a **Distance-Regular Graph** with maximum distance 3.

•*The Colour Palette*: **3 Colours** – Edges (Dist. 1), Pentagonal-Section Diagonals (Dist. 2), and Body Diagonals (Dist. 3).



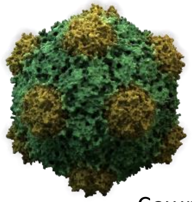
Circogonia icosahedra

🔬 Biological Lens

✨ When they die, their skeletons sink and become **fossils**, preserving their shapes for millions of years!

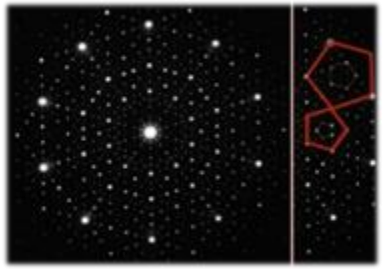
• **Radiolaria** (**Fossils**): Millions of years before Plato, these microscopic marine organisms were already building **silica skeletons** in perfect icosahedral shapes to withstand **deep ocean pressure**.

• **Viral Capsids**: Modern viruses (*Adenovirus*, *HIV*, *Herpes*) use this scheme to build their protective shells with **minimal genetic effort**.



Adenovirus

Cowpea Mosaic Virus



Electron diffraction of Al quasicrystals

🧪 Science & Tech

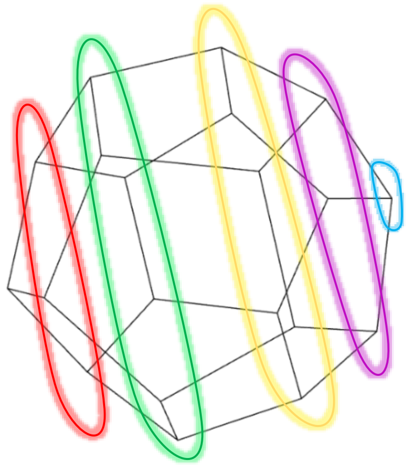
✨ Before this, *Science only believed in repeating patterns*, like tiles on a floor!

• **Aluminium-Manganese Quasicrystals**: These have perfect icosahedral symmetry but **never repeat** – a discovery that rewrote chemistry textbooks and won a **Nobel Prize** (2011).

• **Meteorology**: The **Icosahedral Geodesic Grid** is the global standard to map Earth's atmosphere. It allows meteorologists to predict storms with (almost) **zero geometric distortion**.

Icosahedral Geodesic Grid

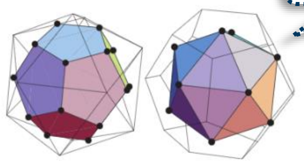
The Dodecahedron: The Complexity of Space



The Dodecahedral Scheme

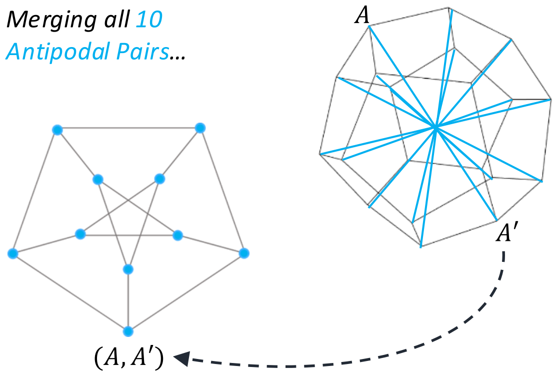
• *The Maths*: The Dodecahedron is a **Distance-Regular Graph** with **maximum distance 5**.

• *The Colours*: It is the most colourful one, with a **5-Colour Palette** to describe its distances – from the Neighbours (Dist. 1) to the single Antipode (Dist. 5).

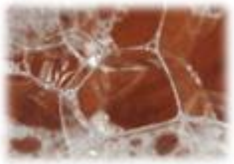


Dodecahedron-Icosahedron Dual Pair

Merging all 10 Antipodal Pairs...



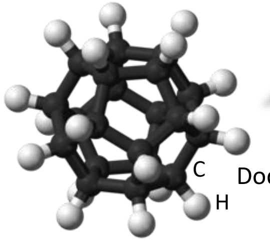
Nature & Cosmos



Dodecahedral Logic in Nature

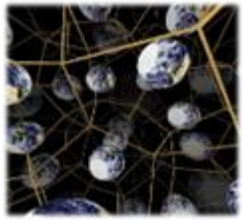
• **Nature's Packing**: When **pomegranate seeds** or **soap bubbles** are packed tightly, they naturally deform into "approximate" dodecahedra to maximize capacity, while minimizing **surface tension**.

• **Poincaré Dodecahedral Space**: Some astrophysicists believe the Universe itself is a finite dodecahedron. Data from the "afterglow" of the **Big Bang** support this thesis.



Dodecahedrane

$C_{20}H_{20}$ is a **Platonic Hydrocarbon**.



PDS Simulation

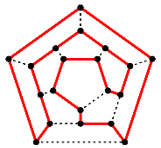
Cosmic Microwave Background: Light traveling through this space would create repeating patterns in the sky – **Hall of Mirrors**.

Archaeology & Game Theory

Roman Dodecahedra and Icosahedron

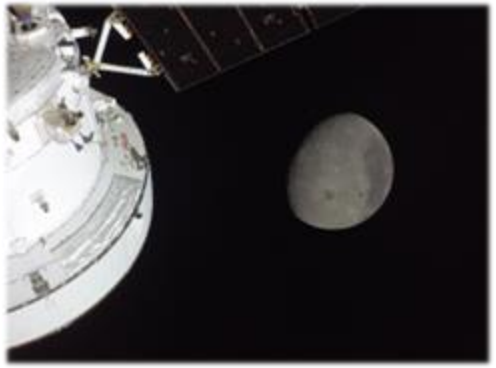
• **Roman Dodecahedra**: Over 100 of these bronze artifacts have been found. No Roman texts mention them. Theories range from **astronomical** and **surveying devices** to simple **knitting tools** .

• **Icosian Game** (Hamilton, 1857). Finding a path through all 20 vertices is the origin of the **Hamiltonian Cycle**, the logic used today by Amazon and Google Maps for delivery routes.



Icosian Game and Hamiltonian Cycle

Final Remarks: The Hidden Colours of Math



Remember these **three lessons** from the world of Association Schemes:

Regularity is Safety

From the Hamming Schemes protecting 5G signals and NASA photos to the DNA codes protecting our life, mathematical regularity is the ultimate **shield against noise and error**.

Symmetry is Efficiency

Nature uses the Platonic Solids because they are the most efficient way to organize matter. Whether it is a virus saving genetic energy or an engineer building a stadium roof, symmetry is the smartest way to **do more with less**.

Maths has Colours

Mathematics is not just black and white numbers. It is a vibrant **palette of relations**. By "colouring" the world through the lens of ASs, we can see the invisible threads that connect a simple gaming die to the shape of the entire Universe.

Thank You!